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TRANSVERSE MERCATOR PROJECTION FOR THE TERRITORY OF MACEDONIA

In this work are given characteristics of Transverse Mercator projection and possibilities for their application like a state's cartographic projection. Then, there are defined the basic elements of projection applicant on territory of Republic of Macedonia. In the end, there are presented some advantages of TM projection beyond Gauss-Kruger's projection like an existing state's cartographic projection.

Keywords: states cartographic projection, Transverse Mercator projection

1. INTRODUCTION

A list of advantages that allow the conformal representation of the Earth's ellipsoid around the circumference of the cylinder, were first discovered by the Dutch cartographer , Gerhard Kremer Mercator in the 16th century.

Figure 1. Gerhard Kremer Mercator – creator of the normal conformal cylindrical cartographic projection

He applied the normal conformal cylindrical projection for the creation of the World's Map for the first time in 1569. This map has been considered as the biggest achievement in the history of carthography and the projection was named Mercator's projection. Some time later, the applicability of transverse cylindrical projections for mapping parts of Earth's surface that have meridian extension was used. More distinguished surveyors and cartographers, such as: Gauss, Cassini, Lambert, Schreiber, Hristov and others worked on developing mathematical expressions for transverse cylindrical projections.

In honor of the great cartographer Mercator, the transverse conformal cylindrical projection is called Transverse Mercator projection. Transverse Mercator projection (together with Lambert conformal conic projection and Stereographic projection) is among the projections that are most often used for geodetic purposes. Over time, the name “geodetic projections” was established for the projections. The basic characteristics are:

- geodetic projections are used to calculate Cartesian coordinates in the plane for points from the geodetic base network, that are set for the needs of surveying a certain territory;
- geodetic projections are used for creation of large and medium scaled topographic plans and topographic maps;
- original surface in these projections is the surface of the reference ellipsoid, which was chosen as the basis of the survey, etc.

Geodetic projections were especially developed in the second half of the 19^{th} and the beginning of the 20^{th} century, with the intensification of the work of the state’s survey in several countries. At the time, Cassini-Soldner equidistant transverse cylindrical projection was especially used. In modern times, the most widely used projection is the Transverse Mercator projection which is accepted as the state cartographic projection in most countries in the world.

2. BASIC CHARACTERISTICS OF THE TRANSVERSE MERCATOR PROJECTION

In the Transverse Mercator projection, the ellipsoid is mapped onto elliptical cylinders, where each cylinder touches the ellipsoid in the middle meridian of the territory that is being mapped.

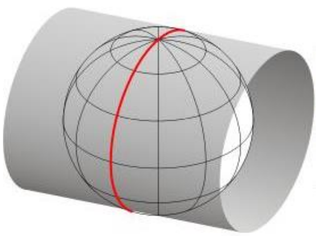

Figure 2. Tangent cylinder of Earth’s ellipsoid

According to the projection’s theory, the mapping is done under the following conditions:

- Conformal projection – with no angle deformations;
- The mean meridian should be mapped as straight line and its projection should have an equal or constant ratio.

According to the characteristics of the projection, in each zone, only the equator and the central meridian are mapped in the plane as straight lines. The other meridians and parallels are mapped in the form of curved lines, symmetrical in relation to the equator and the central meridian.

The Cartesian coordinate system in the Transverse Mercator projection can be set arbitrarily, where the projection of the central meridian defines the X-axis and the projection of the equator defines the Y-axis of the system. The width of the mapped territory in the coordinate system of the Transverse Mercator projection is defined with the maximum allowed linear deformations.

From what has been said so far, it can be concluded that Transverse Mercator projection is basically identical to the Gauss-Kruger projection. The difference between the projections consists of several things about the Transverse Mercator projection:

- Possibility in selection of the mean meridian for the territory that needs to be mapped;
- Possibilty in selection of territory’s width that needs to be mapped in one coordinate system;
- Possibilty for selection of arbitrary module of the linear deformations, whose value does not have to be fixed at -10 cm/km, which makes it possible to obtain better spatial data when mapping the territories with width smaller than 254 km.

Practically, the Gauss-Kruger projection represents a variant of the Transveral Mercator projection with strictly defined rules that enable its universal appllication. However, due to its inflexibility, it is not suitable for mapping small territories, assymetrical in relation to the mean meridian, as is the case with the Republic of Macedonia.

The identical mathematical equations, makes the calculation of the basic dimensions in the Transverse Mercator projection to be made with the known mathematical expressions from the Gauss-Kriger projection.

1. The calculation of the Cartesian coordinates (X, Y) in the plane, for the points on the ellipsoid that are defined by the geographic coordinates φ, λ is done with the following equations:

$$X = \overline{X} + (X_1)\ l^2 + (X_2)\ l^4 + (X_3)\ l^6 \qquad (1)$$

$$Y = (Y_1)\ l + (Y_2)\ l^3 + (Y_3)\ l^5 \qquad (2)$$

$\overline{X}$ - length of meridian arc,
l – difference of longitudes $l = \lambda - \lambda_o$,
(X_1) - (X_3) i (Y_1) - (Y_3) - coefficients.

2. Calculation of latitude and longitude (φ, λ) on the ellipsoid, for points that are defined with Cartesian coordinates (X, Y) in the plane, is done with the following expressions:

$$\varphi = \varphi' + (\varphi_1)\ \overline{Y}^2 + (\varphi_2)\ \overline{Y}^4 + (\varphi_3)\ \overline{Y}^6 \qquad (3)$$

$$l = (l_1)\ \overline{Y} + (l_2)\ \overline{Y}^3 + (l_3)\ \overline{Y}^5 \qquad (4)$$

$\overline{Y}$ - ordinate of a point in relation with the central meridian,
(φ_1) - (φ_3) i (l_1) - (l_3) - coefficients,
φ' – latitude that is obtained in an iterative procedure.

3. Linear deformations in transverse cylindrical projection are calculated with the following expression:

$$\Delta d = d - s = s\,\frac{\overline{Y}_1^2 + 4\overline{Y}_m^2 + \overline{Y}_2^2}{12R_m^2} \qquad (5)$$

d – length of distance in the projection,
s – length of distance on the ellipsoid,
Δd – linear deformation,
$\overline{Y}_1$, $\overline{Y}_2$ - ordinates of endpoints of the line,
$\overline{Y}_m$ - mean ordinate,
R_m – mean curvature radius.

4. The convergence of the meridians at a given point of the projection can be calculated based on the given geographical or Cartesian coordinates for the point.

With known values of geographic coordinates (φ, λ) of a given point in the projection, the convergence of the meridians is calculated with the following expression:

$$c = l\sin\varphi + (c_1)\ l^3 + (c_2)\ l^5 \qquad (6)$$

And if the rectangular coordinates (Y, X) of the point in the projection are known, the convergence of the meridians is calculated with the following expression:

$$c = (c_3)\ \overline{Y} + (c_4)\ \overline{Y}^3 + (c_5)\ \overline{Y}^5 \qquad (7)$$

$\overline{Y}$ - ordinate of a point related to the central meridian,
(c_1) - (c_5) – coefficients,
l – difference of longitudes $l = \lambda - \lambda_o$,

3. ELEMENTS FOR DEFINING THE TRANSVERSE MERCATOR PROJECTION FOR THE TERRITORY OF MACEDONIA

When applying the Transverse Mercator projection, it is necessary to determine the central meridian of the area that is being mapped. The geographic longitude of the central meridian for the territory of our country can be obtained as the arithmetic mean of the geographic longitudes of its extreme points – the easternmost and the westernmost. The easternmost point in the Republic of Macedonia is the border stone num. 47 called Chengino Kale, while the westernmost point is the border stone E16/VIII called Kestenjar. The coordinates of those points are obtained from topographic map TK 25 and those values are:

Table 1. Extreme points of the state's territory

	φ	λ
Chengino Kale	41° 42' 32"	23° 02' 23"
Kestenjar	41° 31' 03"	20° 27' 32"

Arithmetic mean of geographic longitudes is:

$$\lambda_{sr} = \frac{\lambda_{\min} + \lambda_{\max}}{2} = 21^o 44' 57''.5 \qquad (8)$$

In that case, for the mean meridian can be chosen the meridian that have geographic longitudal value of **λ_o = 21° 45'** that is only 2.5" (cca 58 m) far from the ideal mean of the territory of Macedonia.

The Cartesian coordinate system (Y, X), in the Transverse Mercator projection for the Macedonia, is defined by the projections of the equator and the central meridian. Moreover, the projection of the meridian λ_o = 21° 45' defines the X-axis, while the projection of the equator defines the Y-axis of the Cartesian coordinate system.

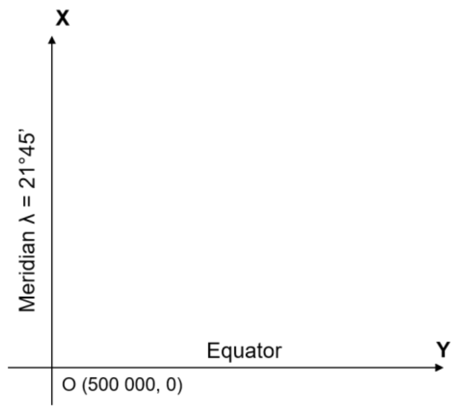

Figure 3. Coordinate system of the Transverse Mercator projection

Taking into account the dimensions of the territory of Macedonia, it is possible to calculate the maximum linear deformations of the projection that will be obtained at easternmost and westernmost points. If the length of the geodetic line is taken as the length of the arc of the parallel between the mean meridian ($\lambda_o = 21^o\ 45'$) and the meridian of the eastern-most and westernmost point (which is about 107.7 km), therefore for the assumed radius of R = 6375 km, the maximum linear deformation is:

$$\Delta d_{max} = 14.3 \text{ cm/km}$$

The accuracy of the projection can be increased by introducing a negative linear deformation (-7 cm/km) on the mean meridian, by introducing the secant cylinder. With the help of the secant cylinder, in the area of 106.67 km east and west of the central meridian, the linear deformations will be less than ± **7 cm/km**. Outside of this area, only a small part of the country's territory will remain where the linear deformations would not be bigger than **+7.3 cm/km**. The inclusion of the linear deformation with value of -7 cm/km is made by multiplying of the Cartesian coordinates ($\overline{Y}, \overline{X}$) by the linear module:

$$m = 0.99993$$

In that way, the modified Cartesian coordinates (Y, X) are calculated with the following expressions:

$$Y = \overline{Y} \cdot 0.99993 + 500000\ m$$
$$X = \overline{X} \cdot 0.99993 \qquad (9)$$

From the expression, it can be seen that, in addition to the reduction, a translational displacement of the coordinate system by 500000 m to the west is performed, thus avoiding negative Y-coordinates (Figure 3).

3.1 LINEAR DEFORMATIONS AND ISOCOLS IN THE TRANSVERSE MERCATOR PROJECTION

Isocols in the Transverse Mercator projection are straight lines that are parallel with the projection of the central meridian that are defined by adequate ordinates $\overline{Y}$.

Isocols for the territory of Macedonia are shown in Table 2.

Table 2. Isocols of the Transverse Mercator projection of Macedonia

Deformations (cm/km)	**Isocols** $\overline{Y}$ (km)
0	0
1	± 28.5
2	± 40.3
3	± 49.4
4	± 57.0
5	± 63.8
6	± 69.8
7	± 75.4
8	± 80.6
9	± 85.5
10	± 90.2
11	± 94.6
12	± 98.8
13	± 102.8
14	± 106.7

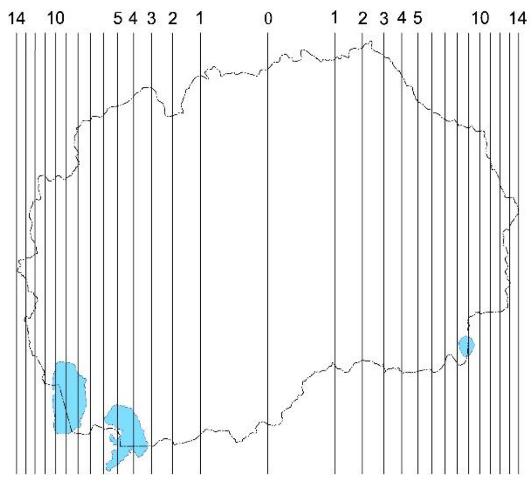

Figure 4. Isocols in the TM projection

After inclusion of the negative linear deformation on the mean meridian of -7 cm/km and reduction of the Cartesian coordinates, the isocols in the TM projection have values in the interval from -7 cm/km to +7.3 cm/km.

Table 3. Isocols of the Transverse Mercator projection after the reduction

Deformations (cm/km)	**Isocols** $\overline{Y}$ (km)
-7	0
-6	± 28.5
-5	± 40.3
-4	± 49.4
-3	± 57.0
-2	± 63.8
-1	± 69.8
0	± 75.4
1	± 80.6
2	± 85.5
3	± 90.2
4	± 94.6
5	± 98.8
6	± 102.8
7	± 106.7

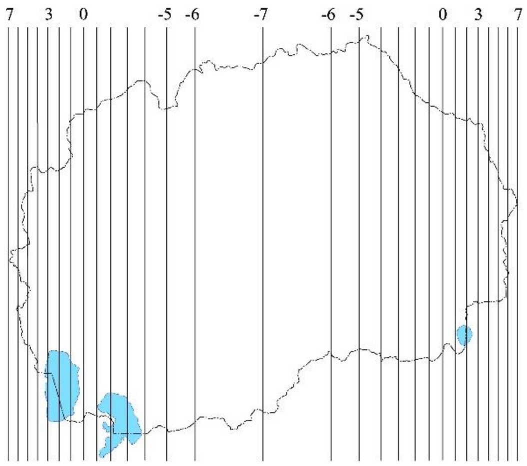

Figure 5. Isocols in TM projection after the reduction

The average linear deformation on the whole territory of Macedonia that is in the Transverse Mercator projection:

$$\Theta = 3.67 \text{ cm/km}$$

After the reduction of the Cartesian coordinates and the introduction of a secant-variant of the TM projection, the average linear deformation gets a value:

$$\Theta = 4.32 \text{ cm/km}$$

3.2 CONVERGENCE OF MERIDIANS

The convergence of the meridian, identically as for the Gauss-Kruger projection, is a quantity that depends of the distance from the meridian of a given point in relation to the central meridian in the projection. Because of the fact that in the Transverse Mercator projection, the X-axis of the coordinate system is identical to the meridian that is near the center of the state's territory, the meridian convergence has almost identical values on both extreme points (easternmost and westernmost) for the territory of Macedonia.

The maximum value of meridian convergence for the easternmost point in the country is:

$$c = +0^{o}\ 51'\ 29.43''$$

4. CONCLUSIONS

On the basis of what has been said above, the basic properties of the Transverse Mercator projection for Macedonia can be sublimated:

- Transverse Mercator projection has the same mathematical expressions as Gauss-Kruger projection;
- The coordinate system of the projection is defined by the projections of the equator and the mean meridian with geographic longitude of $\lambda_o = 21^o\ 45'$;
- Isocols are parallel straight lines, symmetrical in relation to the projection of the central meridian;
- The maximum linear deformation value in the Transverse Mercator projection for Macedonia is *14.3 cm/km;*
- Average linear deformation for the whole country has a value of *3.67 cm/km;*
- For better spatial data quality, the Cartesian coordinates are modulated with constant linear module which has value of m = 0.99993. That means introducing negative linear deformation on the central meridian with the value of -7 cm/km. In that way, the maximum value of linear deformations is being reduced and has a value of *7.3 cm/km;*
- The meridian convergence has a max. value in the easternmost point in the country and its value is: $c_i = +0^o\ 51'\ 29.43''$;
- The distribution of linear deformations for Macedonia is symmetrical and in relation

to the central meridian which geographic longitude is $\lambda = 21^o\ 45'$;

- The projections has detailed mathematical expressions that allow calculation of the geodetic tasks in the plane of the projection;
- More than 80% of the European countries are using a variant of the Transverse Mercator projection for the purpose of state surveying and official cartography.

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