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VERTICAL VIBRATIONS OF RECTANGULAR FLEXIBLE FOUNDATION ON VISCOELASTIC HALFSPACE

This paper presents a novel method, ITM-DSM, for the soil-foundation interaction problems analysis. It is a semi-numerical method based on the coupling of the Integral Transform Method (ITM) and the Dynamic Stiffness Method (DSM). The stiffness matrix of the soil-foundation system is obtained using the substructure technique. The ITM is used to obtain the solution of the wave propagation in the soil, while the DSM is used to calculate the dynamic stiffness matrix of the foundation. Both methods are operating in the frequency domain what makes them suitable for coupling. The number of numerical operations in the frequency domain is reduced by the application of the modal superposition technique. The analysis of vertical vibrations of flexible foundations resting on the viscoelastic half-space is presented. The formulation of the method could be generalized for horizontal and rocking vibrations. It could be also reduced to the problem of flexible strip foundations of infinite length.

Keywords: integral transform method, dynamic stiffness method, fourier transforms, modal superposition method, substructure method

1. INTRODUCTION

The effects of soil-structure interaction (SSI) are not negligible in general [1]. The soil is the integral part of the system and it has an influence on its response. Due to the different nature of the soil and the structure, the SSI problems are usually solved using the substructure technique: the substructures are modeled independently and then coupled applying available boundary conditions.

The modeling of the soil medium is complex in general, but especially in dynamic analysis. The soil must be presented as an unbounded medium so that the energy propagates from the source of vibrations towards the infinity. The traditional modeling techniques, such as the Finite Element Method, cannot address this problem well [2]. That led to the development of new methods such as the

Integral Transform Method (ITM) [3]. The ITM is based on solving Lamé's differential equations of motion, decoupling them using the Helmholtz decomposition and transforming them from partial to ordinary differential equations by a threefold Fourier Transform. The ordinary differential equations are solved in the transformed wavenumber-frequency domain by taking into account the boundary conditions of the system. The solution is transferred in the original space-time domain by threefold inverse Fourier Transform [4].

The SSI problems could be reduced to the soil-foundation interaction (SFI) problems. In the case of a surface, rigid foundation, the problem could be easily solved considering the foundation as a part of the soil surface, and applying kinematic transforms [5]. Since the foundation is always flexible up to a certain level, it is important to investigate the influence of the foundation stiffness on the response of the SFI system. The dynamic response of flexible foundations was investigated mainly by using the boundary element method [6] [7] [8] and finite element method [9] [10] [11]. In this paper a novel ITM-DSM [12] [13] [14] is used to solve that problem.

The DSM [6] is based on the exact solution of the governing differential equations of motion in the space-frequency domain. This results in the exact frequency dependent shape functions of a dynamic stiffness element. The dynamic stiffness matrices of elements are also frequency dependent and can be developed explicitly for one-dimensional beam elements and Levy-type plates. Only one element is sufficient to represent the dynamic behavior at any frequency. In the case of plates with arbitrary boundary conditions, the plate displacements are presented in infinite series form, and the boundary problem is solved using the Projection method [15].

Both ITM and DSM are operating in the frequency domain what makes them suitable for coupling. The coupling of the foundation and the soil is established using the dynamic modal stiffness matrix of the soil. The SFI problem is solved using the modal superposition technique [16].

This paper presents the formulation of the ITM-DSM for rectangular surface foundations. The proposed method is used for obtaining the response of the SFI system in terms of dimensionless displacements in characteristic points of the foundation. The results for displacements have been validated against the existing data from the literature.

2. FORMULATION

The formulation is derived by observing vertical vibrations of a rectangular flexible foundation on the surface (*z=0*) of an elastic, homogeneous and isotropic half-space. It is assumed that the foundation behaves as a Kirchhoff plate. The steady state analysis of the response of the foundation is performed in the frequency domain, (*x, y, ω*). The response of the soil medium is obtained in the wavenumber-frequency domain (k_x, k_y, *z, ω*). It is understood that all functions are *ω* dependent, regarding the steady state analysis. Therefore, the *ω* variable is omitted in the notation of the functions.

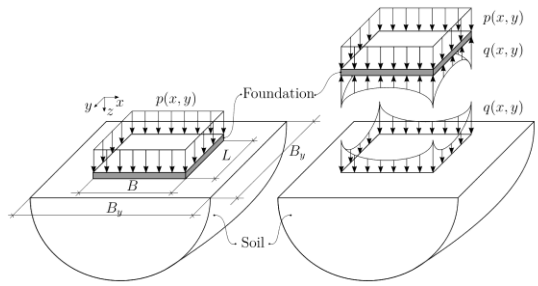

Figure 1. Disposition of the problem

The differential equation of the problem in (*x, y, ω*) domain is given by

$$D\left(\frac{\partial^4 w(x,y)}{\partial x^4}+2\frac{\partial^4 w(x,y)}{\partial x^2 \partial y^2}+\frac{\partial^4 w(x,y)}{\partial y^4}\right)- \\ -\rho h \omega^2 w(x,y)=p(x,y)-q(x,y) \quad (1)$$

where *D* denotes the bending stiffness, *w(x, y)* is the displacement field, *ρ* is the material density and *h* is the thickness of the plate. The bending stiffness of the Kirchhoff plate is defined as

$$D=\frac{Eh^3}{12(1-\nu^2)} \quad (2)$$

where *E* is Young's modulus and *ν* is Poisson's coefficient of the plate. The functions *w(x,y)*, *p(x,y)* and *q(x,y)* are the transverse deflection of the foundation, the vertical load and the soil reaction, respectively. They can be expanded in a series of free vibration modes as follows:

$$w(x,y)=\sum_{n=0}^{N}Y_n\phi_n(x,y)$$
$$p(x,y)=\sum_{n=0}^{N}P_n\phi_n(x,y)\ , \tag{3}$$
$$q(x,y)=\sum_{n=0}^{N}Q_n\phi_n(x,y)$$

where $\phi_n(x)$ represents the orthonormalized mode shape of the foundation for the n^{th} mode and Y_n , P_n and Q_n are modal coefficients:

$$Y_n=\int_{x=0}^{B}\int_{y=0}^{L}w(x,y)\phi_n(x,y)\,dxdy$$
$$P_n=\int_{x=0}^{B}\int_{y=0}^{L}p(x,y)\phi_n(x,y)\,dxdy\ , \tag{4}$$
$$Q_n=\int_{x=0}^{B}\int_{y=0}^{L}q(x,y)\phi_n(x,y)\,dxdy$$

The mode shapes of the foundation are obtained for the case of free vibrations of the completely free plate, solving the eigenvalue problem by using the DSM [17]. The general solution of the problem is of the form

$$\phi_n(x,y)=e^{k_{xn}x}e^{k_{yn}y} \tag{5}$$

where k_{xn} and k_{yn} are wavenumbers in x and y direction, such that

$$k_{xn}^2+k_{yn}^2=\pm\omega_n\sqrt{\frac{\rho h}{D}} \tag{6}$$

The problem is solved by introducing an infinite series of base solution in the (k_x^2, k_y^2) plane [18]. Figure 2 shows the first eight mode shapes of a Kirchhoff plate. The first mode is a translational mode.

Substituting equations (3) and (5) into Eq. (1) gives

$$\sum_{n=0}^{N}(D(k_{xn}^4+2k_{xn}^2k_{yn}{}^2+k_{yn}^4)-\rho h\omega^2)\phi_n(x,y)Y_n=$$
$$=\sum_{n=0}^{N}\phi_n(x,y)P_n-\sum_{n=0}^{N}\phi_n(x,y)Q_n \tag{7}$$

Since mode shapes $\phi_n(x)$ are orthonormal, for a uniform mass distribution, equation (7) can be decoupled into N equations by multiplying with mode shape $\phi_m(x)$ and integrating over the area of the foundation. That gives the system of N equations, written in matrix form:

$$\left[D\left[\mathbf{k^4}\right]-\rho h\omega^2\left[\mathbf{I}\right]\right]\{\mathbf{Y}\}=\{\mathbf{P}\}-\{\mathbf{Q}\} \tag{8}$$

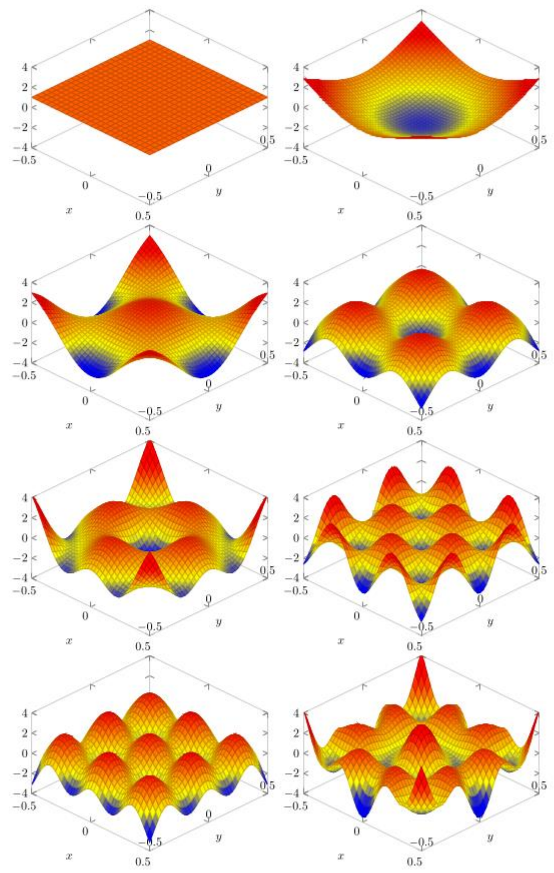

Figure 2. Free vibrations mode shapes of a rectangular foundation for n = 1-8

where {**Y**}, {**P**} and {**Q**} are coefficient vectors of the modal displacement, the load and the soil reaction, respectively, [**I**] is identity matrix and [$\mathbf{k^4}$] is the pure bending wavemode wavenumber matrix of the plate

$$\left[\mathbf{k^4}\right]=\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & \rho h\omega_1^2 & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & \rho h\omega_N^2\end{bmatrix} \tag{9}$$

In equation (9), ω_N are natural frequencies of the plate, obtained using the DSM [19].

Relation between displacements and soil reaction coefficient vectors can be defined as follows

$$[\mathbf{K_s}]\{\mathbf{Y}\}=\{\mathbf{Q}\} \tag{10}$$

Substituting (10) into (8) the equation of motion becomes

$$\left[D\left[\mathbf{k^4}\right]-m_f\omega^2\left[\mathbf{I}\right]+\left[\mathbf{K_S}\right]\right]\{\mathbf{Y}\}=\{\mathbf{P}\} \tag{11}$$

In equations 10 and 11 [$\mathbf{K_S}$] is the modal impedance matrix of the soil, obtained using ITM [14]. Once the modal impedance matrix of the soil is formed, the displacement coefficient vector {**Y**} is obtained by solving the system of equations (10) and finally the displacement field *w(x,y)* is obtained by using equation (3). To obtain the displacement field spectrum, the procedure should be repeated for every frequency in a desired frequency range.

A computer program based on this formulation is developed in MATLAB [20]. The results of the analysis are presented in terms of dimensionless displacements, in order to verify the response of the system with the results from the literature.

3. NUMERICAL EXAMPLE

In this section the vertical displacements of a square, massless, surface foundation excited by a uniformly distributed load are shown, see Figure 1. The damping mechanism is introduced by using a complex modulus with the damping coefficient ξ = 1%. The analysis is performed by taking into account eight shape modes of the foundation shown in Figure 2. Since the problem is axi-symmetrical, only axi-symmetrical mode shapes are used.

The vertical displacement fields of the foundation obtained using the proposed method are compared with the results obtained by Whittaker and Christiano (W&C) [9]. They have modelled the foundation using thin plate finite elements. The displacements of a uniformly loaded plate are observed in three points: centre, edge midway and corner, Figure 3.

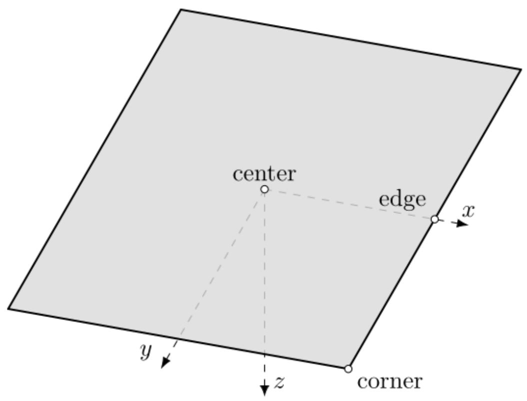

Figure 3. Characteristic points of the foundation

The results are presented in a dimensionless form, $\Delta_i(a_0)$, where Δ_i is the dimensionless vertical displacement at the point *i*

$$\Delta_i = \frac{wG_sB}{(1-\nu_s)\sum F_{ext}} \tag{11}$$

and a_0 is the dimensionless frequency

$$a_0 = \frac{\omega B}{c_s} \tag{12}$$

In equations (11) and (12) *w* is the displacement, G_s is the shear modulus of the soil, ν_s is Poisson's coefficient of the soil, $\sum F_{ext}$ is the resultant of the external force in vertical direction, and c_s is the shear wave velocity in the soil.

The results are obtained for different foundation-soil stiffness ratios *K* introduced by Whittaker and Christiano

$$K = \frac{Eh^3(1-\nu_s^2)}{12(1-\nu^2)G_s\left(\frac{B}{2}\right)^3} \tag{13}$$

where *E* is Young's modulus and ν is Poisson's coefficient of the plate.

Using the proposed method, the analysis is performed for *K* = 0, 0.004, 0.06, 3.3 and the results are shown in Figures 4-7. The case of *K=0* corresponds to the analysis of the soil only (without foundation),. For *K*≥3.3 the foundation is considered rigid.

The results obtained by the proposed method are in a good agreement with the results from the literature. In general, with an increase of the relative stiffness *K* the displacements of the foundation become less spatially dependent. In the frequency range $0<a_0<4$, for *K*=0, the proposed method gives lower amplitudes of the displacement at the centre of the foundation compared to the displacements from literature. For *K*>0, the displacement amplitudes at the centre of the foundation tend to be higher than in literature, as opposed to the displacement amplitudes of the edge point and corner point. The differences between the results increase with the increase of *K* and a_0. In the frequency range $a_0>4$, the highest discrepancies between obtained results and results from literature are observed at the corner of the foundation. This is the point of significant stress concentration that is very difficult to model properly [21].

4. CONCLUSIONS

This paper presents an efficient semi-analytical method for obtaining the dynamic response of rectangular foundations resting on the surface of half-space. The method is based on the modal decomposition. The Dynamic Stiffness Method is used for obtaining mode shapes of the foundation. The impedance matrix of the soil is obtained by using the Integral Transform Method. The comparison of the results obtained using the proposed method with the results from the literature shows that the proposed method provides accurate results with low computational effort. The method could be easily reduced to problems of a strip foundation resting on a homogeneous or horizontally layered half-space as well as extended to the analysis of coupled horizontal and vertical vibrations problems.

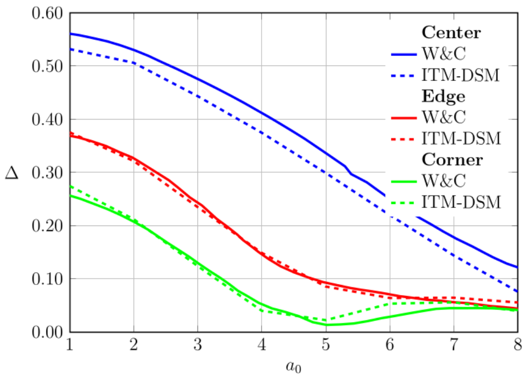

Figure 4. W&C - ITM-DSM comparison of the displacements of the characteristic points of the foundation, K = 0

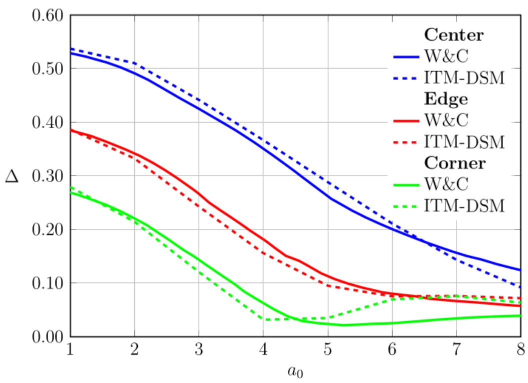

Figure 5. W&C - ITM-DSM comparison of the displacements of the characteristic points of the foundation, K = 0.004

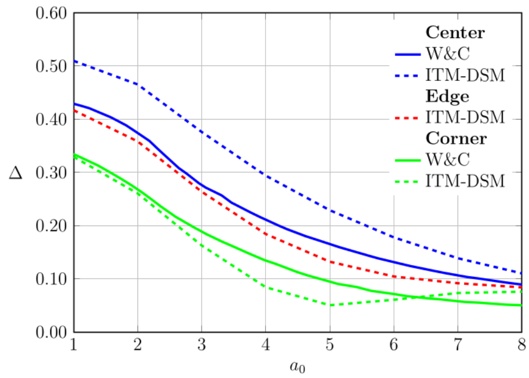

Figure 6. W&C - ITM-DSM comparison of the displacements of the characteristic points of the foundation, K = 0.06

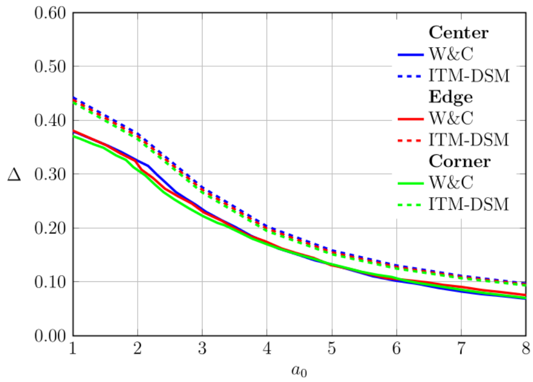

Figure 7. W&C - ITM-DSM comparison of the displacements of the characteristic points of the foundation, K = 3.3

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