TISSOT COMPENSATION PROJECTION FOR THE TERRITORY OF MACEDONIA

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Presenting of Earth area in the plane is always followed with deformation of angles, lengths and areas. Attempting to find cartographic projection for successful minimizing deformation of basic elements was occupation of many cartographs in the past. In this group, the most important is French cartographer M. A. Tissot.

In this work are presented basic characteristic and formulas of Tissot compensation projection. This projection have characteristic for minimizing deformation of angles and lengths, and also have wide usage in cartography especially in the field of states cartographic projection. In common with basic characteristic of projection, in this paper are presented the aspects of use of Tissot compensation projection in the mapping of territory of Republic of Macedonia.

Key words: states cartographic projection, Tissot compensation projection, minimal deformations.

1. INTRODUCTION

Presentation of the curved surface of the Earth on plane is always accompanied by certain deformations of angles, lengths and surfaces. The efforts to find a cartographic projection that can successfully minimize the deformations of the basic elements were the preoccupation of several cartographers. In this respect, the most prominent place belongs to the French cartographer Tissot, who suggested several ways of depicting a part of the Earth's surface on plane from which particular significance has the projection presented in the paper "Memoire sur la représentation des surfaces et les projections des cartes géographiques", which in the literature is known as the Tissue Compensative Projection.

The "compensation" of the projection is due to special concessions referring to the deformations of the angles and the distances, in order to obtain projection of a given territory with an absolutely minimal deformation of angles and distances. Due to this feature, the Tissot projection has a very large application in the cartography, especially in the field of defining of state cartographic projections. In doing so, this projection serves as a benchmark for comparing other projections of the spectrum of possible (conformal) cartographic projections. The Tissot compensatory projection is also significant because of the fact that it answers the question of the size of the territory that can be projected into one coordinate system, that is, in one projection plane, according to a predetermined deformation of lengths.

2. BASIC CHARACTERISTICS OF THE TISSOT PROJECTION

Tissot proposed his projection for mapping territories with relatively small dimensions, i.e. for territories where the differences in the latitudes and longitudes of the extreme points (expressed in absolute measure) are smaller than the unit, taking into account the unit value of the circle arc: $\rho^o = 57^o.29578$, which actually represents the circle arc value of the angle of a radian ($\rho = 1^{rad}$).

The basic task as in all projections is reduces to the definition of the functions f_1 and f_2 which define the analytical relationship between the geographical coordinates of a point on the Earth's ellipsoid and the Cartesian coordinates of the same point in the projection plane.

Instead of geographical coordinates (φ and λ), Tissot introduces s and t, which relate to the auxiliary coordinate system, whose coordinate start is at the central point of the mapped territory. In addition, s represents the length of the arc of the meridian of the middle parallel of the mapped territory by the latitude φ_o, to the parallel across the point T with latitude φ_T, while the coordinate t represents the length of the arc of the middle parallel with the latitude φ_o of the mean meridian with longitude λ_o to the meridian through the point T with latitude λ_T. The coordinates s and t are defined according to known expressions for determining the arc length of a meridian and parallel.By introducing a new coordinate system, the Cartesian coordinates x and y are defined as functions of the coordinates s and t:

$$\begin{aligned} x &= a_1 + a_2 s + a_3 t + a_4 s^2 + a_5 st + \\ & a_6 t^2 + \frac{A}{3}s^3 - B\ s^2 t + C\ st^2 + \frac{D}{3}t^3 + \ldots \\ y &= b_1 + b_2 s + b_3 t + b_4 s^2 + b_5 st + b_6 t^2 \\ & + \frac{A'}{3}s^3 + B' s^2 t - C' st^2 + \frac{D'}{3}t^3 + \ldots \end{aligned} \tag{1}$$

Unknown coefficients (a_1-a_6, b_1-b_6, A, B, C, D, A', B', C' and D') Tissot determined it by introducing additional conditions for the projection, and after satisfying the conditions, the polynomials for x and y continue to retain the character of convergent progression. The mentioned conditions of the projection are as follows:

1. The coordinate start of the ellipsoid has the coordinates φ_o and λ_o, and coincides with the central point of the territory being mapped. It is also the coordinate start of the Cartesian coordinates in the plane. Axes x and y are defined as tangents of the corresponding meridian, or parallel at the coordinate start.
2. The deformations of the lengths at the coordinate start, that is, at the central point of the territory being mapped, should be equal to zero. The coordinate start in all directions (and therefore in the direction of meridian and parallel) is not changed and is equal to the unit.
3. The sizes (1-m) and (1-n), which represent the deformations of the lengths in the direction of the meridian and the parallel, must not contain the members of the first potencies of s and t, and they can differ from each other for the members of the third potencies of s and t.
4. The angle between meridians and parallels in projection can differ from 90^o only for members of the third potencies of s and t.
5. Differences (m-n) and (90-Θ), respectively Θ may not depend on members that contain the second potencies of s and t. Θ represents the deformation of the angle (Θ) that forms the meridians and the parallel in the projection and is defined as:

$$\varepsilon = 90 - \Theta \tag{2}$$

Applying the stated conditions, we arrive at the definitive expressions for determining the Cartesian coordinates (x and y) in the Tissot compensation projection, which according to (*Borčić, 1955*) and (*Stojanovski, 1960*) read:

$$\begin{aligned} x &= s + \frac{\sin\varphi_o}{2r_o}t^2 + \frac{A}{3}s^3 - B\ s^2 t + C\ st^2 + \frac{B}{3}t^3 \\ y &= \frac{r}{r_o}t + \frac{B}{3}s^3 + As^2 t - Bst^2 + \frac{C}{3}t^3 \end{aligned} \tag{3}$$

The terms (3) define the projection in which the deformations of the lengths depend on the second-row members, while the deformations of the angles depend only on the members of the third row. Given the minimal deformations of the angles, the Tissot projection is distinguished by *practical* conformity, and the expression which determines the scales in the direction of the meridian and the parallel is:

$$m = n = 1 + As^2 - 2Bst + \left(\frac{1}{2} - A\right) t^2 \quad (4)$$

The determination of the constants A, B, and C, as well as the coordinates φ_o and λ_o of the coordinate start, are carried out under the condition that the linear deformations (m - 1) are minimal. To this end, the expression (4) gets a new shape:

$$As^2 - 2Bst + \left(\frac{1}{2} - A\right) t^2 - (m-1) = 0 \quad (5)$$

To determine the coefficients A and B, two new variables are introduced -F and ε.

$$\tan\varepsilon = \frac{B}{A - \frac{1}{4}} \qquad F - \frac{1}{4} = \left(A - \frac{1}{4}\right) \sec\varepsilon \quad (6)$$

Angle ε is function of rotation (α) between the coordinate system with axes s and t and the local coordinate system of the ellipse of equal deformations:

$$\varepsilon = 2\ (90^o - \alpha) \quad (7)$$

Figure 1 shows two coordinate systems with a common coordinate start. The coordinate axes of the system (u, v) overlap with the ellipse axes.

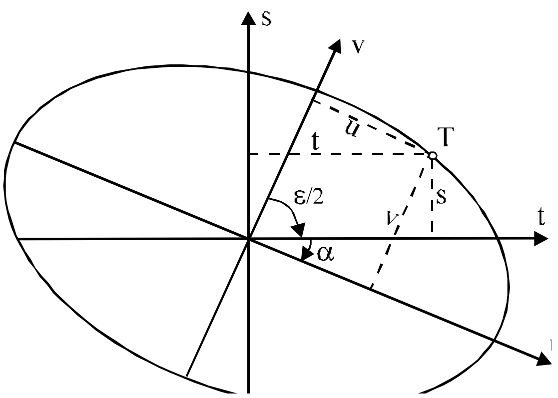

Figure 1. Coordinate system of the ellipse of deformations

To switch from one system to another, it is necessary for the axes s and t to rotate in the direction of the clockwise movement by α.

Examining the characteristics of the projection, Tissot proved two very important properties of the ellipse of deformations.

1. The square of the fourth of the diameter which with the axes of the ellipse (u, v) closes an angle of 45°, is equal to the deformation of the lengths of this projection, that is:

$$\left(\frac{\rho_{45^o}}{4}\right)^2 = m - 1 \quad (8)$$

ρ_{45^o} - the diameter of the ellipse which with the axes (u, v) closes the angle from 45°

2. The square of the axes of the so-called ellipse are inversely proportional to the sizes F and (1/2-F).

Based on the second attribute, the value of the coefficient F is determined according to the expression:

$$F = \frac{b^2}{2\ (a^2 + b^2)} \quad (9)$$

a, b -semi-axes of the ellipse of deformations

The procedure for defining the basic parameters of the Tissot projection (coordinate start and coefficients) is graphic-analytic *(Stojanovski, 1960*). To that end, it is necessary to construct a map in a small scale on the mapped area, on which the distinguished points from the border line are clearly defined.

The map can be made in some projection or constructed on the basis of calculated coordinates (s and t) for the boundary points in relation to the center of the territory with φ_o and λ_o, in which is placed the coordinate system defined by the axes s and t. After the creation of an auxiliary map, a certain number of ellipses with different ratio of their axes a/b are constructed and they are applied to the transparent paper. Thus the constructed ellipses are placed in a row on the map and their rotation and displacement are done until a position is determined which will cover the entire draft of the auxiliary map. For ellipses that cover the entire map (and they touch it in several border points), calculation is performed of their diameter ρ_{45} according to the expression (10) or it can be read graphically.

$$\rho_{45^o} = \frac{2ab\sqrt{2}}{\sqrt{(a^2 + b^2)}} \quad (10)$$

After comparing the obtained results, the ellipse that has a minimum diameter it represents the boundary ellipse. After the *boundary ellipse* is established, the elements are read and calculated for the Tissot projection, which will serve to define the expressions for the Cartesian coordinates x and y, as follows:

- First, geographic coordinates are read (φ_o and λ_o) at the center point of the boundary ellipse relative to the auxiliary map.
- Angle $\varepsilon/2$, which close the semi axis of the boundary ellipse with the coordinate axes s and t, are read directly from the map.
- After determining the rotation angle, according to the expression (9), calculation of the coefficient F is performed.
- Coefficients A and B, as a function F and ε, are calculated according to the expressions:

$$A = \left(F - \frac{1}{4}\right)\cos\varepsilon + \frac{1}{4} \qquad B = \left(F - \frac{1}{4}\right)\sin\varepsilon \quad (11)$$

- Coefficient C is calculated based on the read value of the main parallel φo and the calculated coefficient A, according to the expression:

$$C = \frac{\cos 2\varphi_o}{2\cos^2\varphi_o} - A = \frac{1}{2} - A - \frac{1}{2}\tan^2\varphi_o \quad (12)$$

This procedure defines the basic parameters of the Tissot projection, which can be used to determine the Cartesian coordinates at any point in the mapped territory.

3. ELEMENTS FOR DEFINING THE TISSOT PROJECTION FOR THE TERRITORY OF MACEDONIA

3.1 CONSTRUCTION OF AN AUXILIARY MAP AND DEFINING THE BOUNDARY ELLIPSE OF DEFORMATIONS

The classic approach to the auxiliary map construction, which is used to determine the boundary ellipse in the Tissot projection, requires the choice of a central point in the mapped territory. This point is defined by the coordinates φ_o and λ_o. Then the coordinate differences ($\varphi - \varphi_o$) and ($\lambda - \lambda_o$) are determined between the characteristic boundary points and the center point, and they are shown in a given scale to a paper, giving an approximate image of the mapped area, with precisely defined border points.

The elaboration of this paper avoided the classical approach, and the auxiliary map was obtained by scanning and digitizing cartographic maps in scales 1:200000 (TK 200). Thus, the Macedonia's border line is defined in digital form, with relatively high accuracy and as such it is suitable for further processing and determination of the parameters of the Tissot projection. The auxiliary map is additionally oriented in the existing state coordinate system and georeferenced, thus obtaining a map in the scale of 1:1000000 on which the geographic coordinate network is applied.

On the map are marked 17 characteristic points for which Gauss-Kruger and geographical coordinates are determined (Figure 2). The above points also include the extreme points on the territory of Macedonia - the most northern, the most eastern, the southernmost and the westernmost.

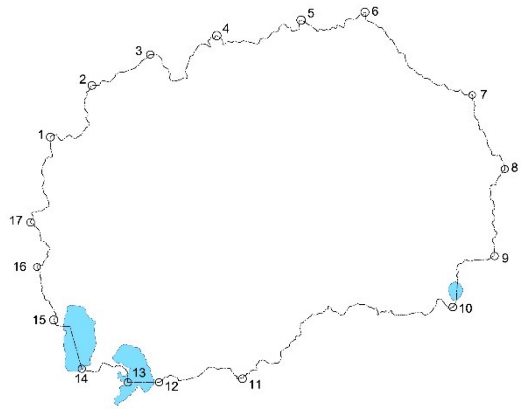

Figure 2. Characteristic borderline points of state territory

On the basis of the so-defined basis, several ellipses have been constructed, which, with the help of translational displacement and angular rotation, are brought into a position to cover the entire state territory and, in addition, to tangent it in several border points. The purpose of the above procedure is to detect the ellipse that best adapts to the shape of the mapped territory, with the diameter of the ellipse, which with its axes occupies an angle of 45°, has a minimal value. This creates a *boundary ellipse* that defines the length deformations in the Tissot projection. To the border ellipse for the territory of the Macedonia, more than 40 trials have been done, some of which whit more characteristic results are presented in tab. 1. The values of the parameters of the test ellipses are expressed in millimeters which because of the scale of the auxiliary map define kilometers in nature.

Table 1. Trial ellipses for determining the boundary ellipse

Trial ellipses				
Point num.	Tangent boundary points	a (mm)	b (mm)	ρ_{45} (mm)
1	6, 8, 14	100.0	98.0	198.0
2	2, 6, 10, 14	125.0	83.5	196.4
3	**2, 6, 10, 13, 14**	**130.0**	**81.5**	**195.3**
4	2, 3, 6, 10, 13	135.0	80.5	195.6
5	3, 6, 10, 11, 13	150.0	78.5	196.7

From the table it can be concluded that the boundary ellipse for the Tissot projection of Macedonia is the ellipse under number 3, which has a minimum diameter:

$$\rho_{45} = 195.3 \text{ mm (km)}$$

and semi-axes:

a = 130.0 mm (km) and **b = 81.5 mm (km).**

After the establishment of the position of the main border ellipse on the auxiliary map, the location of the center from the ellipse was deter-mined, which also represents the coordinate start for Tissot compensation projection of Macedonia. The geographical location at the coordinate start is presented in Figure 3, from which it can be seen that it is located near the village of Izvor, southwest of Veles.

The Gauss-Kruger coordinates of the center are directly read on the screen and they are:

$$Y_o = 7\,562\,050 \text{ m} \qquad X_o = 4\,606\,550 \text{ m}$$

Applying the formulas for the second geodetic task of the Gauss-Kruger projection, the geographical coordinates of the center of the projection can be calculated and they have the following values:

$$\varphi_o = 41^\circ\, 36'\, 00'' \qquad \lambda_o = 21^\circ\, 44'\, 40''$$

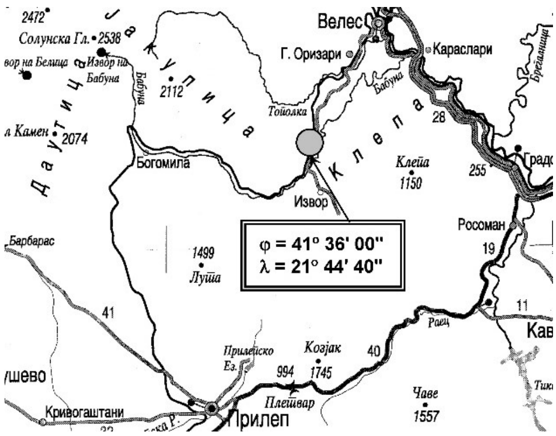

Figure 3. Geographic position of the centre of the projection

Angle of rotation (α) between the axes of the ellipse and the coordinate system (s, t) is graphically read from the auxiliary map and its value is:

$$\alpha = -15^\circ\, 00'\, 45''$$

The position of the boundary ellipse of the Tissot projection and its orientation with respect to the border line from the territory of Macedonia is presented in Figure 4.

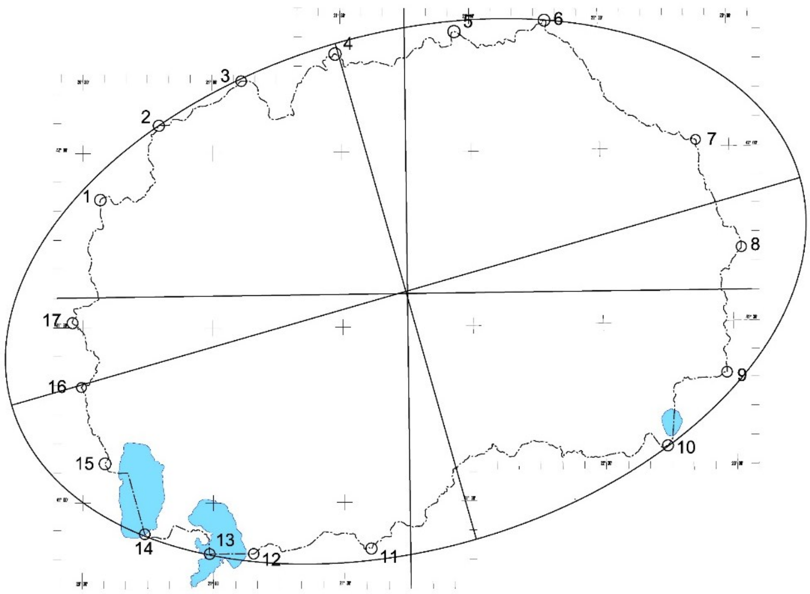

Figure 4. Border ellipse of deformations

3.2 CALCULATING COEFFICIENTS FROM TISSOT COMPENSATION PROJECTION

The values of the angle ε and the coefficient F are determined according to the expressions (7 and 9) are:

$\varepsilon = 210^{o}\ 01'\ 30''$ F = 0.1410708407

The coefficients A and B, as functions of ε and F, are defined by the expressions (11), while the value of the coefficient C is determined by A and the geographic latitude of the projection center (φ_o) according to the expression (12).

A = 0.3443116455 B = 0.05450573606

C = -0.2384429508

3.3 LINEAR DEFORMATIONS AND ISOCOLS IN THE TISSOT PROJECTION

Linear deformations in the Tissot projection can be calculated according to the expression (8) and their value of the boundary ellipse itself is:

$$(m-1) = \left(\frac{\rho_{45^o}}{4}\right)^2 = 2383.9\ \ mm^2(km^2)$$

The linear deformation at any point of the projection is determined by expression (4) and the same expression is also suitable for controlling the calculated coefficients of the projection. In this case, a point that lies on the boundary ellipse is taken, its coordinates s and t are read, and an identical value of the linear deformation with the deformation obtained by expression (8) should be obtained by applying the expression (4).

To obtain the maximum linear deformation, as a relative value with dimensions cm / km, it is necessary to multiply the deformations (m-1) by the expression:

$$\frac{100000}{(MR_o)^2}$$

M - scale of the auxiliary map (1:1000000);

R_o - mean radius of curvature of the center of projection.

The mean radius of curvature in the center of Tissot projection for Macedonia, with latitude $\varphi_o = 41^o\ 36'$ is (*Srbinoski, 2001*):

R_o = 6 374 834.043 m

Thus it comes to the value of maximum linear deformation Tissot compensation projection of Macedonia, which is:

6.0 cm/km

The indicated size is the smallest possible value of the linear deformations that can cover the entire territory of our country *in one coordinate system*.

Lines that merge points with identical linear deformations in the Tissot projection are ellipses, which in the coordinate system (s, t) are defined by the expression (5). The construction of the iscols is facilitated in the coordinate system of the ellipses of deformations with the axes u and v. The axes of this system occupy the angle α with the axes of the system (s, t), and the isocols are defined by the large and small semi-axis, which are obtained according to the following expressions:

$$u = a = \sqrt{\frac{(m-1)}{F}} \qquad v = b = \sqrt{\frac{(m-1)}{\frac{1}{2} - F}} \tag{13}$$

The isocols in the Tissot projection of Macedonia, determined according to the expressions (13), are given in Table 2, and their graphic representation is given in Figure 5.

Table 2. Isocols of the Tissot projection of Macedonia

Deformations (cm/km)	Isocols (ellipse)	
	a (km)	b (km)
0	0	0
1	53.7	33.6
2	75.9	47.6
3	93.0	58.3
4	107.3	67.3
5	120.0	75.2
6	130.0	81.5

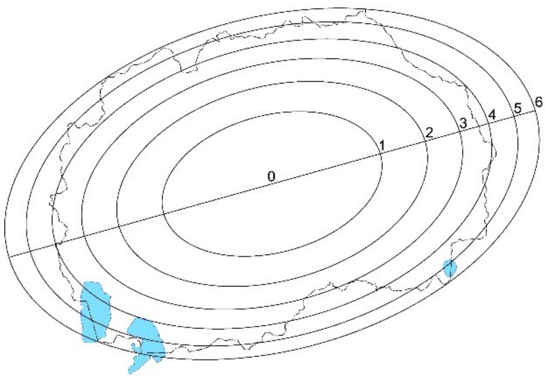

Figure 5. Isocols in the Tissot projection

To reduce the absolute size of the deformations it is necessary to reduce the Cartesian coordinates obtained according to the expressions (3) with a module (m) which is determined according to the expression (*Srbinoski, 2001*):

$$m = 1 - \frac{1}{2}\Delta d_{\max} = 0.99997 \qquad (14)$$

Δd_{max} - maximum linear deformation of the projection

In this way, the maximum linear deformation (by absolute value) is reduced by half, and deformations in the Tissot projection are distributed in the range from **-3 cm / km** to **+3 cm / km.**

The isocols in the Tissot projection defined by their large and small semi axes, after performing the said reduction, are presented in Table 3, and their graphical representation - in Figure 6.

Table 3. Isocols of Tissot compensation projection after the reduction

Deformation (cm/km)	Isocols (ellipse)	
	a (km)	b (km)
-3	0	0
-2	53.7	33.6
-1	75.9	47.6
0	93.0	58.3
1	107.3	67.3
2	120.0	75.2
3	130.0	81.5

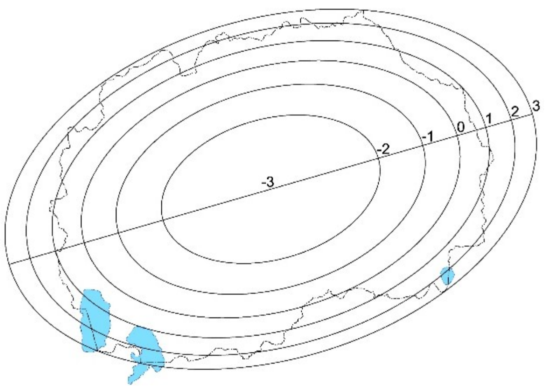

Figure 6. Isocols in Tissot projection after the reduction

The average linear deformation on the whole territory of the Republic of Macedonia is in the Tissot compensation projection:

$$\Theta = 2.34 \text{ cm/km}$$

After the reduction of the Cartesian coordinates and the introduction of a secant-variant of the Tyson projection, the average linear deformation gets a value:

$$\Theta = 1.32 \text{ cm/km}$$

which is an exceptional result that cannot be obtained by applying any other cartographic projection.

5. CONCLUSIONS

On the basis of the above, the basic properties of the Tissot projection for Macedonia can be sublimated:

- Because of its capacity to minimize linear deformations, the Tissot projection is the basis for comparing and evaluating other conformal projections when choosing a state cartographic projection.
- Although the projection in its basis is not conformal, it is characterized by practical conformity (especially when projecting small territories), due to the minimal deformations of angles that depend only on the third movements of the coordinates s and t.
- The boundary ellipse of deformations (for Macedonia) is defined by the parameters:

  a = 130 mm, b = 81.5 mm and

  $\alpha = -15^{\circ}\ 00'\ 45''$

  and the center of the projection, as well as the coordinate start of the Cartesian coordinate system, is located at the point:

  $\varphi_o = 41^{\circ}\ 36'\ 00''$ $\lambda_o = 21^{\circ}\ 44'\ 40''$
- The maximum linear deformation that arises from the shape and dimensions of the boundary ellipse, in the Tissot projecti-on of Macedonia is **6 cm/km**.
- The average linear deformation of the entire state territory is 2.34 cm/km.
- Increasing the accuracy of the projection is achieved by modulating the Cartesian co-ordinates with a module m = 0.99997, i.e. by introducing a negative linear deformati-on of -3 cm/km, which allows the entire territory of Macedonia to be covered by deformations that are not larger from ±3 cm/km.
- After the reduction, the average linear deformation is 1.32 cm/km.

- The layout of the linear deformations is correct, and the isocols have the form of centric ellipses.

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