STEREOGRAPHIC PROJECTION FOR TERRITORY OF THE REPUBLIC OF MACEDONIA

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In stereographic projection, as one of the oldest cartographic projections, Earth is approximate with sphere in the first step and then it is projected on the tangent or cutting plane. The famous Bulgarian geodetic scientist Prof. Vladimir Hristov Ph.D. made stereographic projection with direct projecting of ellipsoid on plane. In this work, basically postulate for composing of stereographic projection for territory of R. of Macedonia are defined.

Keywords: stereographic projection, linear deformations

1. INTRODUCTION

The stereographic projection is one of the oldest known cartographic projections. This projection for the first time was examined by the Greek astronomer Hiparh (180-125 BC), and Ptolemy (87-125 year), who expounded its main features.

The first application of the stereographic projection for the needs of the state survey on the territory of the Balkans dates back to 1863 when it was first used in the regions that belonged to Austria-Hungary. In the aforementioned projection, first, the Earth's ellipsoid was projected onto a sphere, and then a conformal mapping was made of the sphere onto a plane. In addition, a tangent projection plane is applied to the Earth's sphere at one point - approximately in the middle of the mapped territory. As the point of tangency is the trigonometric point of the Gelerthe Hill, near Budapest, which is at the same time a coordinate beginning of the system, named the Budapest coordinate system.

Due to its good character, the stereographic projection falls within the scope of "geodetic" projections, which are often used as national cartographic projections. This is particularly true for countries that have a territory in a circular shape, for which the stereographic projection gives maximum good results.

Due to these characteristics, the stereographic projection is also part of the world projection system where it is used to display the polar regions and is known as the *UPS projection* – (Universal Polar Stereographic Projection).

2. BASIC CHARACTERISTICS OF THE STEREOGRAPHIC PROJECTION

Stereographic perspective azimuthal projections are projections where the mapping is done on a plane that touches or cuts the Earth's sphere (Figure 1). While mapping, the laws of the linear perspective are used, and the observation point V is located on the terrestrial sphere with radius R. The projection's plane, regardless of whether it is tangent or secant, is always normal to the main diameter Qo V.

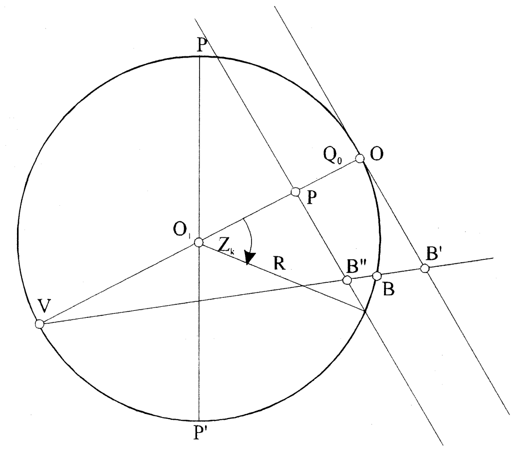

Figure 1. Stereographic projection on tangent, or secant plane

The main diameter in the stereographic projections of the tangent plane is defined by the distance:

$$\overline{VO} = 2R \qquad (1)$$

When a secant plane is applied, corresponding to the main diameter, the distance K is defined, which, according to Figure 1, is:

$$\overline{VP} = K = R(1 + \cos z_K) \qquad (2)$$

Slant stereographic projections on the tangent plane represent the general kind of these projections.

The points on the terrestrial sphere are defined by their sphere-polar coordinates: azimuth (A) and zenith distance (z), relative to the pole of the projection Qo (Figure 2).

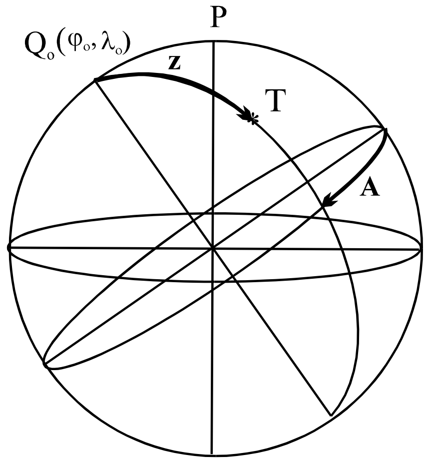

Figure 2. Sphero-polar coordinate system

For the same points in the projection, the polar and the Cartesian coordinates are defined (Figure 3).

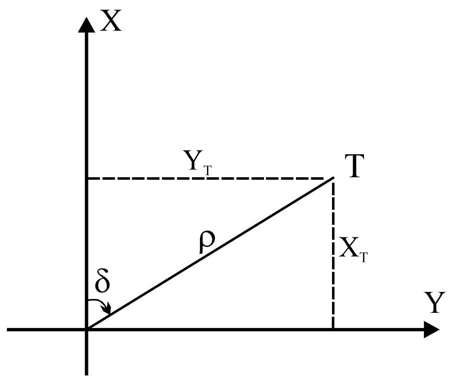

Figure 3. Polar and Cartesian coordinates in the projection plane for point T of the terrestrial sphere

The polar coordinates in the plane (ρ, δ) are defined with the following formulas:

$$\delta = A$$
$$\rho = 2R \cdot tg\frac{z}{2} \qquad (3)$$

The Cartesian coordinates according to (Figure 3) are defined with the following formulas:

$$X = \rho\cos\delta = 2R \cdot tg\frac{z}{2} \cdot \cos A$$
$$Y = \rho\sin\delta = 2R \cdot tg\frac{z}{2} \cdot \sin A \qquad (4)$$

These coordinates can also be expressed in function of the geographical coordinates of point T and the pole Qo:

$$X = \frac{2R\ (\sin\varphi\cos\varphi_0 - \cos\varphi\sin\varphi_0\cos l)}{1 + (\sin\varphi\sin\varphi_0 + \cos\varphi\cos\varphi_0\cos l)} \qquad (5)$$

$$Y = \frac{2R\cos\varphi\sin l}{1 + (\sin\varphi\sin\varphi_0 + \cos\varphi\cos\varphi_0\cos l)}$$

Where φ_0 is the latitude of the pole Qo, φ is the latitude of the point T, and l is the difference between the longitude λ at point T and the longitude λ_0 of the coordinate beginning Qo, i.e. : $l = \lambda - \lambda_0$.

The linear deformation module in the direction of the vertical (μ_1) and the almucantar (μ_2) is defined by the formula:

$$\mu_1 = \mu_2 = \frac{1}{\cos^2 \frac{z}{2}} > 0 \qquad (6)$$

For stereographic projections on secant plane, the linear deformations for points above the secant plane are negative, while for points below the projection plane they are positive. In both cases, the absolute values of the linear deformations grow with distancing from the intersecting circle, called the "zero deformation circuit" (Fig. 4)

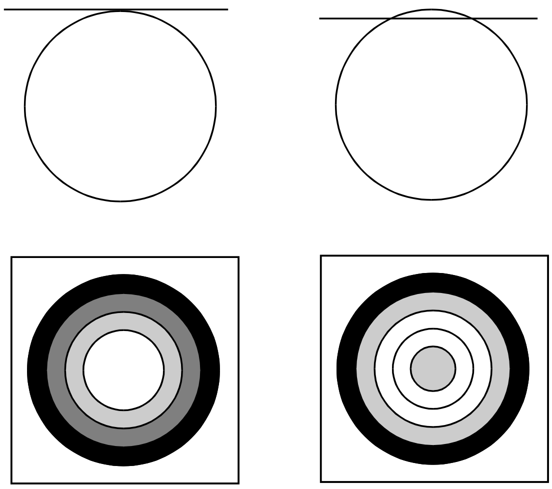

Figure 4. Schematic representation of deformations in the stereographic projection on the tangent and on the secant plane

3. STEREOGRAPHIC PROJECTION ACCORDING TO THE METHOD OF PROF. VLADIMIR HRISTOV

The stereographic perspective azimuthal projection, developed according to the method with constant coefficients (from Prof. Dr. Vladimir Hristov), defines the following geometric characteristics:

- The Earth's ellipsoid is approximated with a sphere.
- The point Qo is the center of the mapped territory, and at the same time it is the point where the projection plane touches the Earth's sphere (Figure 5):

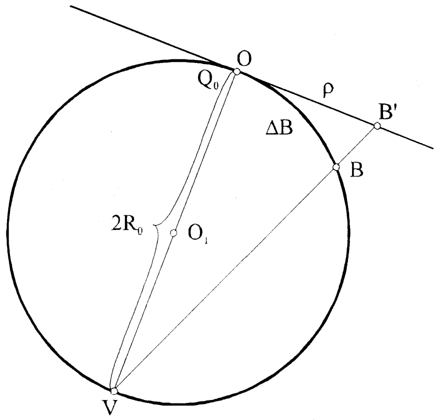

Figure 5. Basic geometric characteristics on the stereographic projection of prof. V. Hristov

- The perspective projection of all points of the Earth's sphere is performed from point V which is found on the opposite side of the Earth's surface.
- The projection is defined as conformal, which means that there are linear and surface deformations, while there are no angular deformations.
- For a coordinate beginning in the projection plane, the point O is adopted, which is the image of the center of the projection Qo. Point O is a coordinate beginning in the system of cartesian stereographic coordinates.
- A meridian with geographical longitude λ_0, which passes down the center of the projection Qo, is mapped as a straight line that is adopted for the X-axis, with a positive direction north. The normal direction to X-axis at Point O is defined as Y-axis in the projection plane.

Having in mind the aforementioned conditions, the expressions for the basic calculations in the stereographic projection can be made.

3.1 DIRECT TASK

The transformation of the geographical coordinates (φ, λ) into Cartesian stereographic coordinates (Y, X) is done by means of expressions with constant coefficients, which depend on the difference in geographical latitudes (Δφ) and the difference on the geographical longitudes (*l*) between the individual projected points and the center of the projection Qo (φ_o, λ_o).

Expressions for calculation are obtained by the method of prof. V. Hristov, according to which all the elements dependent on the geographical latitude φ (such as: the isometric latitude q, the radius of curvature of the parallels r, etc.) are developed in the Taylor series near the center of the projection Qo. The resulting differentials, calculated around the central point, represent constants that are grouped together to give constant coefficients.

In relation of the fact that the stereographic projection is defined as conformal, for this projection are valid Cauchy–Riemann's differential equations that say:

$$x + iy = f(q + il) \quad (7)$$

After the necessary development and differentiation, as well as the separation of the real from the imaginary part, the final expressions for calculating the cartesian coordinates X and Y are obtained:

$$\begin{aligned} X = {} & a_{10}\,\Delta\varphi + a_{20}\,\Delta\varphi^{2} + \dots + a_{70}\,\Delta\varphi^{7} + \\ & + a_{02}\,l^{2} + a_{12}\,\Delta\varphi\, l^{2} + \dots + a_{52}\,\Delta\varphi^{5}\, l^{2} + \\ & + a_{04}\,l^{4} + a_{14}\,\Delta\varphi\, l^{4} + \dots + a_{34}\,\Delta\varphi^{3}\, l^{4} + \\ & + a_{06}\,l^{6} + a_{16}\,\Delta\varphi\, l^{6} + \dots \end{aligned} \quad (8)$$

$$\begin{aligned} Y = {} & b_{01}\,l + b_{11}\,\Delta\varphi\, l + \dots + b_{61}\,\Delta\varphi^{6}\, l + \\ & + b_{03}\,l^{3} + b_{13}\,\Delta\varphi\, l^{3} + \dots + b_{43}\,\Delta\varphi^{4}\, l^{3} + \\ & + b_{05}\,l^{5} + b_{15}\,\Delta\varphi\, l^{5} + \dots \end{aligned} \quad (9)$$

In the expressions (8) and (9) the coefficients a_{ij} and b_{ij} represent the sum of products of the coefficients a_n and c_n, defined by the expressions:

$$a_n = \frac{1}{n!}\left(\frac{d^n x}{dq^n}\right)_0 = \frac{1}{n!} f^n(q)_0 \quad (10)$$

$$c_n = \frac{1}{n!}\left(\frac{d^n q}{d\varphi^n}\right)_0 = \frac{1}{n!} f^n(\varphi)_0 \quad (11)$$

The calculation of the coordinates X and Y is most easily performed in a matrix form.

3.2 INDIRECT TASK

The calculation of the second geodetic task, i.e. the calculation of the geographical coordinates (φ, λ) on the rotational ellipsoid, from the given Cartesian stereographic coordinates (Y, X), are also expressed in terms of special constant coefficients. In this case, the coordinate differences Δφ and Δλ are calculated, in relation to the geographical coordinates φ_o, λ_o) and *l*, for the center of the projection Qo. The geographical coordinates φ and λ are determined according to the following formulas:

$$\varphi = \varphi_0 + \Delta\varphi \qquad \lambda = \lambda_0 + l \quad (12)$$

In view of the fact that the conditions for conformal mapping apply, the Cauchy-Riemann differential equation for this transformation is:

$$\Delta q + il = F(x + iy) \quad (13)$$

After the development of the equation in Taylor series, as well as the separation of the real of the imaginary part, the finite expressions are obtained for Δφ and *l*, which have the form:

$$\begin{aligned} \Delta\varphi = {} & A_{10}\,x + A_{20}\,x^{2} + \dots + A_{50}\,x^{5} + \\ & + A_{02}\,y^{2} + A_{12}\,x\,y^{2} + \dots + A_{42}\,x^{4}\,y^{2} + \\ & + A_{04}\,y^{4} + A_{14}\,x\,y^{4} + \dots + A_{06}\,y^{6} + \dots \end{aligned} \quad (14)$$

$$\begin{aligned} l = {} & B_{01}\,y + B_{11}\,x\,y + \dots + B_{51}\,x^{5}\,y + \\ & + B_{03}\,y^{3} + B_{13}\,x\,y^{3} + \dots + B_{33}\,x^{3}\,y^{3} + \\ & + B_{05}\,y^{5} + B_{15}\,x\,y^{5} + \dots \end{aligned} \quad (15)$$

The coefficients A_{ij} and B_{ij} in the expressions (14) and (15) represent the sum of products with different degrees of coefficients A_n and C_n, defined by the formulas:

$$A_n = \frac{1}{n!}\left(\frac{d^n q}{dx^n}\right)_0 \quad (16)$$

$$C_n = \frac{1}{n!}\left(\frac{d^n \varphi}{dq^n}\right)_0 \quad (17)$$

3.3 LINEAR DEFORMATIONS AND DEFORMATIONS OF LENGHT

The stereographic projection, defined as a conformal projection, possesses the properties that the angles of the terrestrial sphere don't deform in the projections, while the lengths and the surfaces of the mapping are deformed. The linear deformation modulus (μ) in stereographic projections is defined with the formula:

$$\mu = \frac{1}{\cos^2 \frac{z}{2}} \qquad (18)$$

Figure 6 shows the dependence of the deformation modulus μ from the zenith distance z, the arc L, and the radius Ro to the terrestrial sphere.

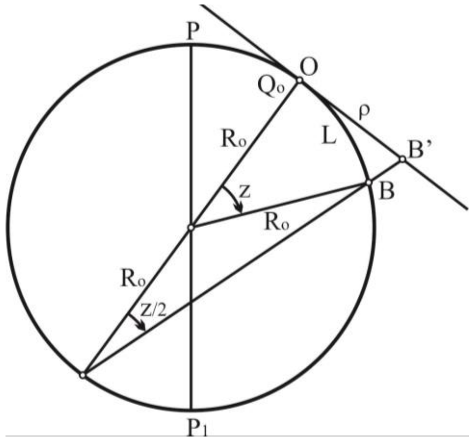

Figure 6. Dependence of the deformation modulus μ from the zenith distance, arc L and the radius R_o on the terrestrial sphere

From the picture is obtained:

$$\frac{z}{2} = \frac{L}{2R_0} \qquad (19)$$

Having this in mind, according to a series of mathematical operations, the linear module of deformations in a finite form reads as follows:

$$\mu = 1 + \frac{L^2}{4R_0^2} \qquad (20)$$

The definitive expression for the calculation of the relative linear deformations (D_t), according to the expression (20), takes the form:

$$D_T = \mu - 1 = \frac{L^2}{4R_0^2} \qquad (21)$$

$$D_T = \mu - 1 = \frac{\rho^2}{4R_0^2} = \frac{X^2 + Y^2}{4R_0^2} \qquad (22)$$

In the expression (22) X and Y are the Cartesian stereographic coordinates of point B (Figure 6), for which the linear deformations are calculated.

4. ELEMENTS FOR DEFINING THE STEREOGRAPHIC PROJECTION FOR THE TERRITORY OF THE REPUBLIC OF MACEDONIA

4.1 DEFINING THE CENTER OF THE PROJECTION Q_O

For the definition of the center of the projection (Qo) it is necessary to construct a circle with a minimum radius that will tangent the territory of the Republic of Macedonia in a few extreme points. This is for locating the center of the projection in the middle of the territory that is being mapped, in order to obtain minimal deformation of the lengths in the projection. For this purpose, the auxiliary map in the scale of 1: 1000000 was used and the extreme points defined during the reconstruction of the Tissot's projection in the Republic of Macedonia.

From such a defined basis, the construction of circles, which with the help of translation shifting, is brought in a position to tangent the border line from the territory of the Republic of Macedonia. The circle with a minimal radius touches the boundary line in points 8, 14 and 17 and runs close to points 1, 6, 7, 15 and 16 (Figure 7). The radius of the circle is read from the screen:

ρ_{min} = 108.0 mm (km)

After the establishment of the border circle on the digital basis, the location of the center of circle was determined, which is a coordinate beginning for the stereographic projection of the Republic of Macedonia.

The geographical coordinates of the center of the projection have the following values:

$\varphi_o = 41^o\ 30'\ 30''$ $\lambda_o = 21^o\ 45'\ 50''$

Figure 7 shows the position of the boundary circle and the center of the stereographic projection in relation to the border line from the territory of R. Macedonia.

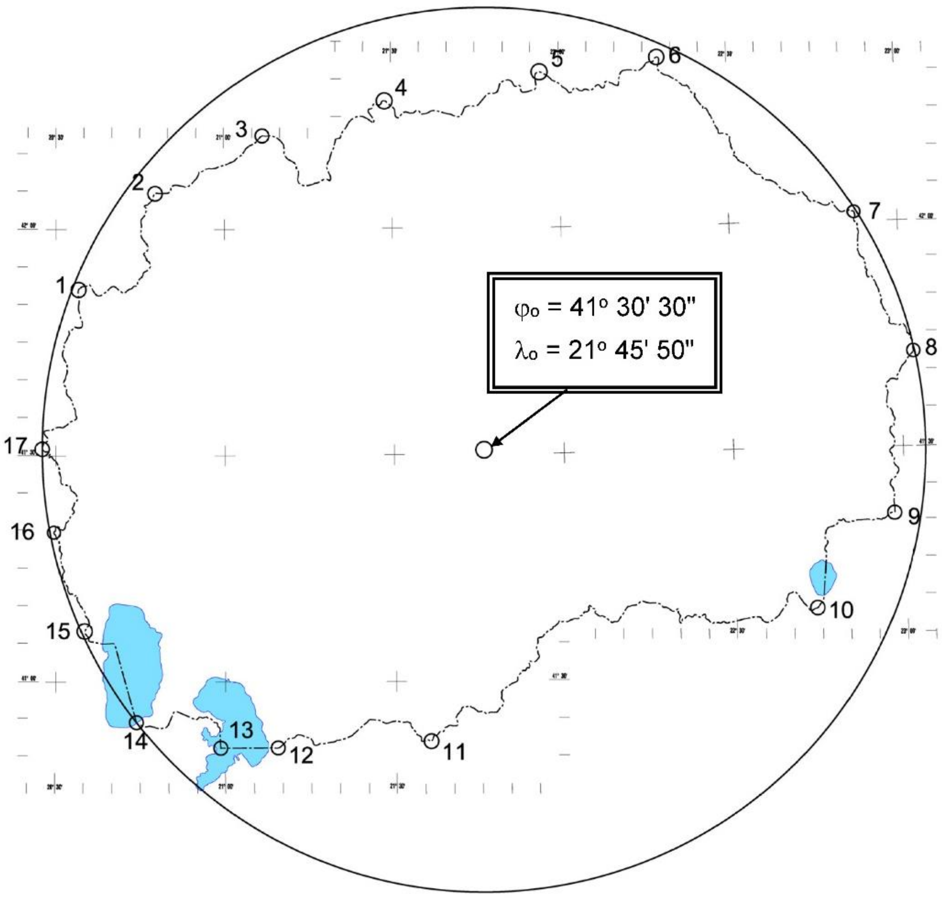

Figure 7. The boundary circle and the coordinate beginning of the stereographic projection of the Republic of Macedonia

4.2 LINEAR DEFORMATIONS AND DISTORTION ISOGRAMS IN THE STEREOGRAPHIC PROJECTION OF THE REPUBLIC OF MACEDONIA

The calculation of the linear deformations is the function of the radius of the circle (the distance relative to the center of the projection). Knowing the radius of the boundary circle, it is possible to determine the maximum linear deformations in the stereographic projection of Republic of Macedonia.

The average radius of curvature at the center of the stereographic projection with a geographic latitude $\varphi_o = 41^o\ 30'\ 30''$, is:

$$R_o = 6374766.265 \text{ m}$$

and the maximum value of the linear deformations is:

$$\mu - 1 = \frac{\rho^2}{4R_0^2} 100000 = 7.18 \text{ cm/km}$$

The value of the linear deformations (7.2 cm / km) can be controlled by calculating the stereographic coordinates for the points in which the circle touches the boundary line. In this way, the following linear deformations in points 8, 14 and 17 are obtained.

Point No.	deformation (cm/km)
8	7.26
14	7.25
17	7.30

The minimum difference of approximately 1 mm/km between exactly calculated and graphically defined linear deformations is due to the limited graphic accuracy of the auxiliary map.

After the completed control, the maximum value of the linear deformations in the stereographic projection of Republic of Macedonia with the application of the tangent projection plane was determined:

7.3 cm/km

The distortion isograms in the stereographic projection are circles in the middle of the central point of the projection. They are defined with the appropriate radius values that can be obtained by transforming the formula (22):

$$\rho = 2R_0 \sqrt{\frac{(\mu - 1)}{100000}} \tag{23}$$

The distortion isograms in the stereographic projection of Republic of Macedonia are presented in Table 1.

Table 1. Distortion isograms in the stereographic projection of Republic Macedonia

deformation (cm/km)	**distortion isograms (circle)**
	ρ (km)
0	0
1	40.3
2	57.0
3	69.8
4	80.6
5	90.2
6	98.8
7	106.7

The graphical representation of the distortion isograms is given in Figure 8

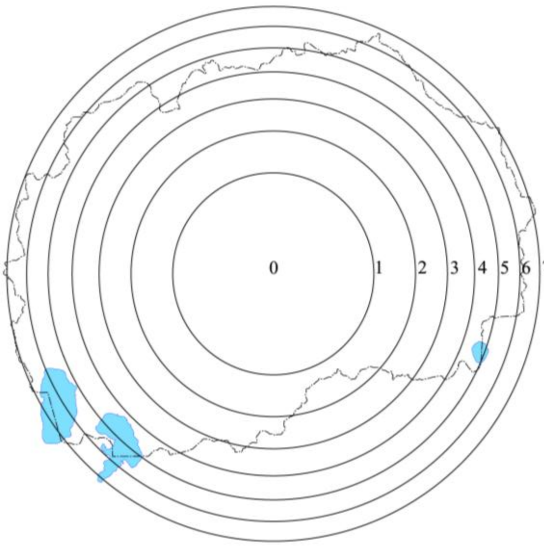

Figure 8. Distortion isograms in the stereographic projection of the Republic of Macedonia

The accuracy of the projection can be increased by introducing a negative linear deformation (-3.5 cm/km), or by introducing a secant projection plane. In this way, the maximum linear deformation is reduced (by absolute value) to half, and the deformations in the stereographic projection are distributed in the range of **-3.5 cm/km to +3.8 cm/km.**

The radius values (ρ), that define the distortion isograms after the reduction are presented in Table 2.

Table 2. Distortion isograms in the stereographic projection after the reduction

deformation (cm/km)	**distortion isograms (circle)**
	ρ (km)
-3.5	0
-3	28.5
-2	49.4
-1	63.7
0	75.4
1	85.5
2	94.6
3	102.8

The graphical representation of the distortion isograms after the reduction is given in Figure 9.

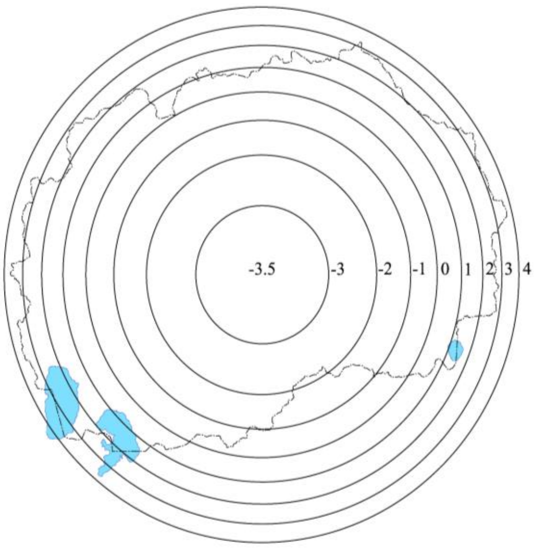

Figure 9. Distortion isograms in the stereographic projection on R. of Macedonia after the reduction

Based on the above, the basic properties of the stereographic projection for Republic of Macedonia can be sublimated:

- The maximum linear deformation, which arises from the dimensions of the boundary circle, is 7.3 cm / km in the stereographic projection of Republic of Macedonia.

- Increasing the accuracy of the projection is achieved by modulating the Cartesian coordinates with the module m = 0.999965, or introducing a negative linear deformation of -3.5 cm / km, which enables the entire territory of Republic of Macedonia to be covered with deformations that don't exceeding ±3.8 cm/km.
- The distortions of lengths are 75% smaller than the distortions in actual Gaus-Kruger projection.
- The shape of the territory of the Republic of Macedonia is very suitable for use of stereographic projection.
- The linear deformation layout is normal, and the isocols have the shape of a circle.
- There are no angle distortions.

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